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A COCOMPLETE BUT NOT COMPLETE ABELIAN CATEGORY

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Abstract. An example of a cocomplete abelian category that is not complete is constructed.

1. Introduction

In this paper we answer a question that was posted on the internet site MathOverflow by Simone Virili [Vir12]. The question asked whether there is an abelian category that is cocomplete but not complete. It seemed that there must be a standard example, or an easy answer, but despite receiving a fair amount of knowledgeable interest, after many months the question still had no solution.

Recall that a complete category is one with all small limits, and that for an abelian category, which by definition has kernels, completeness is equivalent to the existence of all small products. Similarly, because of the existence of cokernels, cocompleteness for abelian categories is equivalent to the existence of all small coproducts.

A common, but wrong, first reaction to the question is that there are easy natural examples, with something such as the category of torsion abelian groups coming to mind: an infinite direct sum of torsion groups is torsion, but an infinite direct product of torsion groups may not be. Nevertheless, this category does have products: the product of a set of groups is simply the torsion subgroup of their direct product. In categorical terms, the category of torsion abelian groups is a coreflective subcategory of the category of all abelian groups (which certainly has products), with the functor sending a group to its torsion subgroup being right adjoint to the inclusion functor.

Similar, but more sophisticated, considerations doom many approaches to finding an example.

Recall that an (`AB5`) category, in the terminology of Grothendieck [Gro57], is a cocomplete abelian category in which filtered colimits are exact, and that a Grothendieck category is an (`AB5`) abelian category with a generator. The favourite cocomplete abelian category of the typical person tends to be a Grothendieck category, such as a module category or a category of sheaves, but it is well known that every Grothendieck category is complete (for a proof see, for example, [KS06, Prop. 8.3.27]).

More generally, any locally presentable category is complete [AR94, Cor. 1.28], and most standard constructions of categories preserve local presentability.

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2. The construction

First we shall fix a chain of fields $\{k_\alpha \mid \alpha \in \mathbf{On}\}$ indexed by the ordinals such that k_β/k_α is an infinite degree field extension whenever $\alpha < \beta$. For example, we could take a class $\{x_\alpha \mid \alpha \in \mathbf{On}\}$ of variables indexed by $\mathbf{On}$ and let $k_\alpha = \mathbb{Q}(X_{<\alpha})$ be the field of rational functions in the set of variables $X_{<\alpha} = \{x_\gamma \mid \gamma < \alpha\}$.

We generally adopt the convention that categories are locally small (i.e., the class of morphisms between two objects is always a set). However, we'll start by defining a "category" $\mathbf{C}$ which is not locally small.

An object V of $\mathbf{C}$ consists of a k_α-vector space V_α for each ordinal α, together with a k_α-linear map $v_{\alpha,\beta} : V_\alpha \to V_\beta$ for each pair $\alpha < \beta$ of ordinals, such that $v_{\alpha,\gamma} = v_{\beta,\gamma} v_{\alpha,\beta}$ whenever $\alpha < \beta < \gamma$. (When we denote an object by an upper case letter such as V, we will always use, without further comment, the corresponding lower case letter for the linear maps $v_{\alpha,\beta}$.)

A morphism $\varphi : V \to W$ of $\mathbf{C}$ consists of a k_α-linear map $\varphi_\alpha : V_\alpha \to W_\alpha$ for each ordinal α, such that $\varphi_\beta v_{\alpha,\beta} = w_{\alpha,\beta}\varphi_\alpha$ whenever $\alpha < \beta$.

Composition is defined in the obvious way. If $\vartheta : U \to V$ and $\varphi : V \to W$ are morphisms, then $(\varphi\vartheta)_\alpha = \varphi_\alpha \vartheta_\alpha$.

It is straightforward to check that $\mathbf{C}$ is an additive (in fact, k_0-linear) category, but not locally small: for example, if V is the object with $V_\alpha = k_\alpha$ for every α and $v_{\alpha,\beta} = 0$ for all $\alpha < \beta$, then a morphism $\varphi : V \to V$ has $v_\alpha : k_\alpha \to k_\alpha$ multiplication by some scalar $\lambda_\alpha \in k_\alpha$ for some arbitrary choice of $\{\lambda_\alpha \mid \alpha \in \mathbf{On}\}$, so the class of endomorphisms of V is a proper class.

Proposition 2.1. $\mathbf{C}$ *is a (not locally small) abelian category with (small) products and coproducts in which (small) filtered colimits are exact.*

Proof. It is straightforward to check that the obvious "pointwise" constructions give kernels, cokernels, products and coproducts. For example, if $\varphi : V \to W$ is a morphism, then the kernel of φ is the object U with U_α the kernel of $\varphi_\alpha : V_\alpha \to W_\alpha$ and $u_{\alpha,\beta} : U_\alpha \to U_\beta$ the natural map between the kernels of φ_α and φ_β induced by $v_{\alpha,\beta}$.

Since all these constructions are "pointwise", it is also straightforward to check that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel (so that $\mathbf{C}$ is abelian), and that small filtered colimits are exact, as the verification reduces to the corresponding facts for the category of k_α-vector spaces. □

The (locally small) category that we're really interested in is a full subcategory $\mathbf{G}$ of $\mathbf{C}$.

Definition 2.2. *An object V of $\mathbf{C}$ is* α-**grounded** *if, for every $\beta > \alpha$, V_β is generated as a k_β-vector space by the image of $v_{\alpha,\beta}$. The full subcategory of $\mathbf{C}$ consisting of the α-grounded objects is denoted by α-$\mathbf{G}$.*

Definition 2.3. *An object V of $\mathbf{C}$ is* **grounded** *if it is α-grounded for some ordinal α. The full subcategory of $\mathbf{C}$ consisting of the grounded objects is denoted by $\mathbf{G}$.*

Theorem 2.4. $\mathbf{G}$ *is a (locally small)* (`AB5`) *abelian category that is not complete.*

Proof. Let V be an α-grounded object. If $\varphi : V \to W$ is a morphism then, for $\beta > \alpha$, φ_β is determined by φ_α, and so the class of morphisms $\varphi : V \to W$ is a set. Hence $\mathbf{G}$ is locally small.

Let $\varphi : V \to W$ be a morphism between α-grounded objects. Clearly the kernel and cokernel of φ are also α-grounded. Hence $\mathbf{G}$ is closed under kernels and cokernels in $\mathbf{C}$ and so it is an exact abelian subcategory of $\mathbf{C}$.

If $\{V^i \mid i \in I\}$ is a set of grounded objects, then there is some ordinal α so that every V^i is α-grounded. Then $\bigoplus_{i\in I} V^i$ is also α-grounded. Thus $\mathbf{G}$ is cocomplete and the inclusion functor $\mathbf{G} \hookrightarrow \mathbf{C}$ preserves coproducts, and hence all colimits. Exactness of filtered colimits in $\mathbf{G}$ therefore follows from the same property for $\mathbf{C}$.

Thus $\mathbf{G}$ is a locally small (`AB5`) abelian category.

For an ordinal α, let M^α be the object with

$$M^\alpha_\beta = \begin{cases} 0 & \text{if } \beta < \alpha \\ k_\beta & \text{if } \beta \geq \alpha, \end{cases}$$

and $m^\alpha_{\beta,\gamma} : k_\beta \to k_\gamma$ the inclusion map for $\alpha \leq \beta < \gamma$. Then M^α is α-grounded.

If W is any object of $\mathbf{C}$ then a morphism $\varphi : M^\alpha \to W$ is determined by $\varphi_\alpha : M^\alpha_\alpha = k_\alpha \to W_\alpha$, so $\mathbf{C}(M^\alpha, W) \cong W_\alpha$ and M^α represents the functor $W \mapsto W_\alpha$ from $\mathbf{C}$ to k_α-vector spaces. Also, if $\alpha < \beta$ then the obvious morphism $\varphi : M^\beta \to M^\alpha$, with φ_γ the identity map $k_\gamma \to k_\gamma$ for $\gamma \geq \beta$, induces a commutative square

$$\begin{tikzcd} \mathbf{C}(M^\alpha, W) \arrow[r, "\sim"] \arrow[d, "\mathbf{C}(\varphi,W)"'] & W_\alpha \arrow[d, "w_{\alpha,\beta}"] \\ \mathbf{C}(M^\beta, W) \arrow[r, "\sim"] & W_\beta \end{tikzcd}$$

We shall show that in $\mathbf{G}$ there is no product of a countably infinite collection of copies of M^α, so $\mathbf{G}$ is not complete.

Suppose that P is the product (in $\mathbf{G}$) of countably many copies of M^α. Then for any ordinal $\beta \geq \alpha$,

$$\mathbf{G}(M^\beta, P) \cong \mathbf{G}(M^\beta, M^\alpha)^{\mathbb{N}} \cong (M^\alpha_\beta)^{\mathbb{N}} \cong k_\beta^{\mathbb{N}}$$

as k_β-vector spaces.

But if $\alpha < \beta < \gamma$ then $k_\gamma^{\mathbb{N}}$ is not generated as a k_γ-vector space by $k_\beta^{\mathbb{N}}$, since k_γ / k_β is an infinite field extension. So P cannot be β-grounded for any β. □

Remark 2.5. *We have already noted that products do exist in* $\mathbf{C}$*, with the obvious pointwise construction. However, the product of* α*-grounded objects need not be* β*-grounded for any* β*.*

Products also exist in α-$\mathbf{G}$*, since although the pointwise product may not be* α*-grounded,* α-$\mathbf{G}$ *is a coreflective subcategory of* $\mathbf{C}$*. We can* α*-ground an object* V *of* $\mathbf{C}$ *by replacing* V_β *by* $v_{\alpha,\beta}(V_\alpha)$ *for* $\beta > \alpha$*, and the product of a set of objects in* α-$\mathbf{G}$ *is obtained by* α*-grounding the product in* $\mathbf{C}$*.*

Remark 2.6. *Each category α-$\mathbf{G}$ is a cocomplete and complete abelian category (in fact, a Grothendieck category, with $\bigoplus_{\beta\leq\alpha} M^\beta$ a generator). The inclusion functors α-$\mathbf{G} \to \beta$-$\mathbf{G}$ are exact and preserve coproducts, but do not preserve products, which explains why their union $\mathbf{G}$ has coproducts but does not have products (or at least not in an obvious way). Thanks to Zhen Lin Low for making this observation in a comment on MathOverflow* [Vir12].

Remark 2.7. *Of course, an example of a complete abelian category that is not cocomplete can be constructed by taking the opposite category of $\mathbf{G}$.*

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